The Theory of Electrodynamics in a Linear Dielectric

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We adopt the continuum limit of a linear, isotropic, homogeneous, transparent, dispersion-negligible dielectric of refractive index n and examine the consequences of the effective speed of light in a stationary dielectric, c/n, for D'Alembert's principle and the Lagrange equations. The principles of dynamics in the dielectric-filled space are then applied to the electromagnetic Lagrangian and we derive equations of motion for the macroscopic fields. A direct derivation of the total energy–momentum tensor from the field strength tensor for the electromagnetic field in a dielectric is used to demonstrate the utility of the new theory by resolving the century-old Abraham–Minkowski electromagnetic momentum controversy in a way that preserves the principles of conservation of energy, conservation of linear momentum, and conservation of angular momentum.

I. INTRODUCTION

In the real-world, a material is composed of microscopic particles embedded in the vacuum. The characteristics of the material are determined by the types of particles and the manner of their interactions. Because an explicit accounting of all of the particles and their interactions is problematic for most materials, we are usually content with a continuum description of material effects in terms of macroscopic parameters that are proportional to the number density of the microscopic particles in a suitably large volume. Although the macroscopic treatment does not have the same physical content of the microscopic theory, it must nevertheless be a self-contained and self-consistent formalism of the physical processes that occur in the limited system.

In continuum electrodynamics, electromagnetic fields are analyzed using an empirical set of equations of motion for the fields, the macroscopic Maxwell equations, in which a simple dielectric is treated as a region of space where a macroscopic polarization field exists in response to the presence of a macroscopic electric field. Alternatively, we can view a stationary dielectric as a continuous homogeneous region of space in which light travels at a reduced speed, c/n, compared to the speed of light c in the vacuum. In this article, we derive a self-contained and self-consistent theoretical treatment of classical continuum electrodynamics from this fundamental property of a macroscopic dielectric. The significance of the new continuum electrodynamics is that the four-dimensional formulation produces the correct traceless symmetric total energy–momentum four-tensor [1–4] that embodies, in continuum form, the laws of conservation of energy, conservation of linear momentum, and conservation of angular momentum.

We proceed as follows: In section II, we adopt the continuum limit of a stationary linear dielectric of refractive index n and examine the consequences of the effective light speed c/n for D'Alembert's principle and the Lagrange equations. In Section III, we derive equations of motion for the macroscopic fields in a stationary dielectric medium

$$\nabla\times\mathbf{B}+\frac{n}{c}\frac{\partial\mathbf{\Pi}}{\partial t}=\frac{n\mathbf{J}}{c} \tag{1.1a}$$

$$\nabla\times\mathbf{\Pi}-\frac{n}{c}\frac{\partial\mathbf{B}}{\partial t}=\frac{\nabla n}{n}\times\mathbf{\Pi} \tag{1.1b}$$

$$\nabla\cdot\mathbf{B}=0 \tag{1.1c}$$

$$\nabla\cdot\mathbf{\Pi}=-\frac{\nabla n}{n}\cdot\mathbf{\Pi}-\rho \tag{1.1d}$$

from the Lagrangian. Here, $\mathbf{B}=\nabla\times\mathbf{A}$ is the magnetic field, $\mathbf{\Pi}=(n/c)\partial\mathbf{A}/(\partial t)$ is the conjugate momentum field, ρ is the total charge density, $\mathbf{J}$ is the free charge current, and $\mathbf{A}$ is the vector potential.

There is no question that the classical macroscopic Maxwell equations

$$\nabla\times\mathbf{H}-\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}=\frac{\mathbf{J}}{c} \tag{1.2a}$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \tag{1.2b}$$

$$\nabla\cdot\mathbf{B}=0 \tag{1.2c}$$

$$\nabla\cdot\mathbf{D}=\rho \tag{1.2d}$$

successfully explain the phenomena of classical continuum electrodynamics, with the notable exception of the century-old Abraham–Minkowski momentum controversy. The same record of experimental and theoretical validation largely applies to the set of macroscopic electrodynamic equations of motion, Eqs. (1.1a)–(1.1d). Apart from the scaling of the free charge current, each of the equations of motion for the macroscopic fields in continuum electrodynamics, Eqs. (1.1a)–(1.1d), is mathematically equivalent to a corresponding

Maxwell-Heaviside equation, Eqs. (1.2a)–(1.2d) [5, 6]. However, the transformations do not comprise a tensor transformation and the two sets of coupled equations of motion are not equivalent. In Section IV, we develop the tensor form of continuum electrodynamics. We construct the field-strength tensor and derive the total energy–momentum tensor, a result that has been sought for over a century [7–11]. We discuss the content and role of the total energy–momentum tensor in terms of the laws of conservation of energy, conservation of linear momentum, and conservation of angular momentum in a continuum.

II. PARTICLE DYNAMICS IN A DIELECTRIC FILLED SPACE

We consider an arbitrarily large region of space to be filled with a linear, isotropic, homogeneous, transparent dielectric in a regime in which dispersion, electrostriction, and magnetostriction are negligible and, for convenience, we apply the term simple linear dielectric to this medium. In the rest frame of the simple linear medium, the constant refractive index n is the only property of a linear dielectric that is significant to the current problem. Let the rest frame of the dielectric be $S(t,x,y,z)$ with orthogonal axes x, y, and z. Then position vectors in S are denoted by $\mathbf{x}=(x,y,z)$. If a light pulse is emitted from the origin at time $t=0$, then

$$x^2+y^2+z^2-\left(\frac{ct}{n}\right)^2=0 \tag{2.1}$$

describes wavefronts in the S system. Writing time as a spatial coordinate $\bar{x}_0=ct/n$, the four-vector $(\bar{x}_0,\mathbf{x})=(ct/n,x,y,z)$ represents the position of a point as a matter of geometry [12]. Because we are using an effective speed of light in defining our timelike coordinate $\bar{x}_0$, the macroscopic theory is not, and should not be expected to be, Lorentz invariant. Lorentz invariance is tied to the special theory of relativity and the microscopic Maxwell equations for fields in a vacuum. A microscopic theory of a dielectric is always possible and such a theory will be Lorentz invariant as light travels at speed c between scattering events. However, Lorentz invariance is not an intrinsic symmetry of a continuous medium in which the electromagnetic field has been averaged over multiple scattering events creating a macroscopic field that travels with an effective speed that is less than c [12–14].

For a system of particles, the transformation of the position vector $\mathbf{x}_i$ of the i^{th} particle to J independent generalized coordinates is

$$\mathbf{x}_i=\mathbf{x}_i(\tau;q_1,q_2,\ldots,q_J), \tag{2.2}$$

where $\tau=t/n$. Applying the chain rule, we obtain the virtual displacement

$$\delta\mathbf{x}_i=\sum_{j=1}^{J}\frac{\partial\mathbf{x}_i}{\partial q_j}\delta q_j \tag{2.3}$$

and the velocity

$$\mathbf{u}_i=\frac{d\mathbf{x}_i}{d\tau}=\sum_{j=1}^{J}\frac{\partial\mathbf{x}_i}{\partial q_j}\frac{dq_j}{d\tau}+\frac{\partial\mathbf{x}_i}{\partial\tau} \tag{2.4}$$

of the i^{th} particle in the new coordinate system. Substitution of

$$\frac{\partial\mathbf{u}_i}{\partial(dq_j/d\tau)}=\frac{\partial\mathbf{x}_i}{\partial q_j} \tag{2.5}$$

into the identity

$$\frac{d}{d\tau}\left(m\mathbf{u}_i\cdot\frac{\partial\mathbf{x}_i}{\partial q_j}\right)=m\frac{d\mathbf{u}_i}{d\tau}\cdot\frac{\partial\mathbf{x}_i}{\partial q_j}+m\mathbf{u}_i\cdot\frac{d}{d\tau}\left(\frac{\partial\mathbf{x}_i}{\partial q_j}\right) \tag{2.6}$$

yields

$$\frac{d\mathbf{p}_i}{d\tau}\cdot\frac{\partial\mathbf{x}_i}{\partial q_j}=\frac{d}{d\tau}\left(\frac{\partial}{\partial(dq_j/d\tau)}\frac{1}{2}m\mathbf{u}_i^2\right)-\frac{\partial}{\partial q_j}\left(\frac{1}{2}m\mathbf{u}_i^2\right). \tag{2.7}$$

For a system of particles in equilibrium, the virtual work of the applied forces $\mathbf{f}_i$ vanishes and the virtual work on each particle vanishes leading to the principle of virtual work

$$\sum_i\mathbf{f}_i\cdot\delta\mathbf{x}_i=0 \tag{2.8}$$

and D'Alembert's principle

$$\sum_i\left(\mathbf{f}_i-\frac{d\mathbf{p}_i}{d\tau}\right)\cdot\delta\mathbf{x}_i=0. \tag{2.9}$$

Using Eqs. (2.3) and (2.7) and the kinetic energy of the i^{th} particle

$$T_i=\frac{1}{2}m\mathbf{u}_i^2, \tag{2.10}$$

we can write D'Alembert's principle, Eq. (2.9), as

$$\sum_j^J\left[\left(\frac{d}{d\tau}\left(\frac{\partial T}{\partial(dq_j/d\tau)}\right)-\frac{\partial T}{\partial q_j}\right)-Q_j\right]\delta q_j=0 \tag{2.11}$$

in terms of the generalized forces

$$Q_j=\sum_i\mathbf{f}_i\cdot\frac{\partial\mathbf{x}_i}{\partial q_j}. \tag{2.12}$$

If the generalized forces come from a generalized scalar potential function V [15], then we can write Lagrange equations of motion

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial(\partial q_j/\partial\tau)}\right)-\frac{\partial L}{\partial q_j}=0, \tag{2.13}$$

where $L=T-V$ is the Lagrangian. The canonical momentum is therefore

$$p_j=\frac{\partial L}{\partial(dq_j/d\tau)} \tag{2.14}$$

in a linear medium. Comparable derivations for the vacuum case appear in, for example, Goldstein [15] and Marion [16].

III. MACROSCOPIC EQUATIONS OF MOTION FOR FIELDS IN A DIELECTRIC

We consider a dielectric block illuminated at normal incidence from the vacuum by a quasimonochromatic electromagnetic pulse in the plane-wave limit. The simple dielectric medium is linear, isotropic, homogeneous, transparent, and dispersionless. Although dielectrics in the real world are much more complicated than this model of a simple linear dielectric, theoretical physics encourages reducing the complexity of the real world and eliminating non-essential details in order to determine what is truly important. In particular, temporal dispersion is inconsequential for the arbitrarily long quasimonochromatic electromagnetic field that is considered here. The dielectric block is draped with a gradient-index antireflection coating and spatial variation of the refractive index is sufficiently smooth that reflection, and the associated radiation pressure, can be neglected. Then in the rest frame of the dielectric block, the refractive index is a smoothly varying, real, and time-independent function of position in a large, but finite, region of space.

The field theory [17, 18] is based on a generalization of the discrete case in which the dynamics are derived from a Lagrangian density $\mathcal{L}$. The generalization of the Lagrange equation, Eq. (2.13), for fields in a linear medium is [17, 18]

$$\frac{d}{d\bar{x}_0}\frac{\partial\mathcal{L}}{\partial(\partial A_j/\partial\bar{x}_0)}=\frac{\partial\mathcal{L}}{\partial A_j}-\sum_i\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i A_j)}, \tag{3.1}$$

where $\bar{x}_0=ct/n$ is the time-like coordinate in the material and x_1, x_2, and x_3 correspond to the respective x, y and z coordinates. We adopt the typical conventions that Roman indices run from one to three, Greek indices run from zero to three, and ∂_i represents the operator $\partial/\partial x_i$. We take the Lagrangian density of the electromagnetic field in the medium to be

$$\mathcal{L}=\frac{1}{2}\left(\left(\frac{\partial\mathbf{A}}{\partial\bar{x}_0}\right)^2-(\nabla\times\mathbf{A})^2\right)+\frac{n\mathbf{J}}{c}\cdot\mathbf{A}. \tag{3.2}$$

Evaluating the components of Eqs. (3.1), we have

$$\frac{\partial\mathcal{L}}{\partial(\partial A_j/\partial\bar{x}_0)}=\frac{\partial A_j}{\partial\bar{x}_0} \tag{3.3}$$

$$\frac{\partial\mathcal{L}}{\partial A_j}=\frac{nJ_j}{c} \tag{3.4}$$

$$\sum_i\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i A_j)}=[\nabla\times\nabla\times\mathbf{A}]_j \tag{3.5}$$

for the Lagrangian density given in Eq. (3.2). Substituting the individual terms, Eqs. (3.3)–(3.5), into Eq. (3.1), the Lagrange equations of motion for the electromagnetic field in a dielectric are the three orthogonal components of the vector wave equation

$$\nabla\times\nabla\times\mathbf{A}+\frac{\partial^2\mathbf{A}}{\partial\bar{x}_0^2}=\frac{n\mathbf{J}}{c}. \tag{3.6}$$

For fields, the canonical momentum density

$$\Pi_j=\frac{\partial\mathcal{L}}{\partial(\partial A_j/\partial\bar{x}_0)} \tag{3.7}$$

supplants the discrete canonical momentum defined in Eq. (2.14). We can write the second-order equation, Eq. (3.6), as a set of first-order differential equations. To that end, we introduce macroscopic field variables

$$\mathbf{\Pi}=\frac{\partial\mathbf{A}}{\partial\bar{x}_0} \tag{3.8}$$

$$\mathbf{B}=\nabla\times\mathbf{A}. \tag{3.9}$$

Obviously, $\mathbf{\Pi}$ is the canonical momentum field density whose components were defined in Eq. (3.7) after making the substitutions indicated by Eq. (3.3). Substituting the definition of the canonical momentum field $\mathbf{\Pi}$, Eq. (3.8), and the definition of the magnetic field $\mathbf{B}$, Eq. (3.9), into Eq. (3.6), we obtain a Maxwell–Ampère-like law

$$\nabla\times\mathbf{B}+\frac{\partial\mathbf{\Pi}}{\partial\bar{x}_0}=\frac{n\mathbf{J}}{c}\,. \tag{3.10}$$

The divergence of $\mathbf{B}$, Eq. (3.9), and the curl of $\mathbf{\Pi}$, Eq. (3.8), produce Thompson's Law

$$\nabla\cdot\mathbf{B}=0 \tag{3.11}$$

and a Faraday-like law

$$\nabla\times\mathbf{\Pi}-\frac{\partial\mathbf{B}}{\partial\bar{x}_0}=\frac{\nabla n}{n}\times\mathbf{\Pi}\,, \tag{3.12}$$

respectively. We posit the charge continuity law

$$\frac{\partial\rho_f}{\partial\bar{x}_0}=-\nabla\cdot\frac{n\mathbf{J}}{c} \tag{3.13}$$

that corresponds to conservation of free charges with a free charge density ρ_f in the continuum limit. (Simply multiplying the vacuum charge continuity law by n results in a discrepancy between the divergence of the variant Maxwell–Ampère law, Eq. (3.12) and the temporal derivative of the Gauss-like law, Eq. (3.16).) The divergence of the variant Maxwell–Ampère Law, Eq. (3.10),

$$\frac{\partial}{\partial\bar{x}_0}\nabla\cdot\mathbf{\Pi}=-\frac{\nabla n}{n}\cdot\frac{\partial\mathbf{\Pi}}{\partial\bar{x}_0}+\nabla\cdot\frac{n\mathbf{J}}{c} \tag{3.14}$$

is combined with the charge continuity law, Eq. (3.13), to obtain

$$\frac{\partial}{\partial\bar{x}_0}\nabla\cdot\mathbf{\Pi}=-\frac{\nabla n}{n}\cdot\frac{\partial\mathbf{\Pi}}{\partial\bar{x}_0}-\frac{\partial\rho_f}{\partial\bar{x}_0}. \tag{3.15}$$

Integrating Eq. (3.15) with respect to the temporal coordinate yields a version of Gauss's law

$$\nabla \cdot \mathbf{\Pi} = -\frac{\nabla n}{n} \cdot \mathbf{\Pi} - \rho_f - \rho_b, \tag{3.16}$$

where ρ_b is a constant of integration corresponding to a bound charge density. This completes the set of first-order equations of motion for the macroscopic fields, Eqs. (3.10)–(3.12) and (3.16) that were introduced in Sec. I as Eqs. (1.1a)–(1.1d).

IV. FIELD AND ENERGY–MOMENTUM TENSORS

In the Maxwell–Heaviside formulation of classical continuum electrodynamics, there are two pairs of fields, $\{\mathbf{E}, \mathbf{B}\}$ and $\{\mathbf{D}, \mathbf{H}\}$, and two field-strength tensors. Here, there is a single pair of fields $\{\mathbf{\Pi}, \mathbf{B}\}$ and a single field-strength tensor. The field-strength tensor,

$$F^{\alpha\beta} = \begin{bmatrix} 0 & \Pi_x & \Pi_y & \Pi_z \\ -\Pi_x & 0 & -B_z & B_y \\ -\Pi_y & B_z & 0 & -B_x \\ -\Pi_z & -B_y & B_x & 0 \end{bmatrix}, \tag{4.1}$$

is obtained in the usual way from

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{4.2}$$

for homogeneous materials.

The reduction to a single field-strength tensor and a single pair of fields results in an elegant simplification of four-dimensional continuum electrodynamics. For example, the total energy–momentum tensor is defined in terms of the field tensor by [19, 20]

$$T^{\alpha\beta} = -F^{\alpha\lambda} F^\beta_\lambda + \frac{1}{4} g^{\alpha\beta} F_{\lambda\nu} F^{\lambda\nu}, \tag{4.3}$$

such that [1–4]

$$T^{\alpha\beta} = \begin{bmatrix} (\mathbf{\Pi}^2 + \mathbf{B}^2)/2 & (\mathbf{B}\times\mathbf{\Pi})_x & (\mathbf{B}\times\mathbf{\Pi})_y & (\mathbf{B}\times\mathbf{\Pi})_z \\ (\mathbf{B}\times\mathbf{\Pi})_x & W_{11} & W_{12} & W_{13} \\ (\mathbf{B}\times\mathbf{\Pi})_y & W_{21} & W_{22} & W_{23} \\ (\mathbf{B}\times\mathbf{\Pi})_z & W_{31} & W_{32} & W_{33} \end{bmatrix}, \tag{4.4}$$

where

$$W_{ij} = -\Pi_j \Pi_k - B_j B_k + \frac{1}{2}(\Pi^2 + B^2)\delta_{ij} \tag{4.5}$$

is the Maxwell stress tensor and $g^{\alpha\beta}$ is the diagonal metric tensor with non-zero elements $g^{00} = 1$ and $g^{ii} = -1$.

The form of the energy–momentum tensor has been debated for over a century [7–11]. The best known candidates are the 1908 Minkowski [21] tensor and the 1909 Abraham [22] tensor. However, neither the Minkoswski momentum nor the Abraham momentum is conserved. It has been proven that the Gordon momentum [7, 23]

$$G_G = \int_\sigma \frac{\mathbf{B}\times\mathbf{\Pi}}{c}\, dv, \tag{4.6}$$

is conserved in our closed system consisting of a homogeneous dielectric illuminated by a quasimonochromatic pulse at normal incidence through a gradient-index antireflection coating. [1–4]. The total energy

$$U = \int_\sigma \frac{1}{2}\left(\mathbf{\Pi}^2 + \mathbf{B}^2\right) dv \tag{4.7}$$

is likewise conserved. Then, in the absence of sources or sinks, the conserved quantities

$$U = \int_\sigma T^{00}\, dv \tag{4.8a}$$

$$P^i = \frac{1}{c}\int_\sigma T^{i0}\, dv \tag{4.8b}$$

are temporally invariant [19, 20].

The homogeneous tensor continuity equation is descriptive of energy and momentum conservation in an unimpeded flow. The four-divergence operator for a system with a position four-vector $(\bar{x}_0, x, y, z)$ is [1–4]

$$\bar{\partial}_\beta = \left(\frac{n}{c}\frac{\partial}{\partial t}, \partial_x, \partial_y, \partial_z\right). \tag{4.9}$$

Then the electromagnetic continuity equations

$$\frac{\partial}{\partial \bar{x}_0}\left[\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2)\right] + \nabla\cdot(\mathbf{B}\times\mathbf{\Pi}) = 0 \tag{4.10a}$$

$$\frac{\partial}{\partial \bar{x}_0}(\mathbf{B}\times\mathbf{\Pi}) + \nabla\cdot\mathbf{W} = \mathbf{0} \tag{4.10b}$$

are the components of the homogeneous tensor continuity equation

$$\bar{\partial}_\beta T^{\alpha\beta} = 0 \tag{4.11}$$

applied to the total energy–momentum tensor, Eq. (4.4). Therefore, for a continuous dielectric without charges or currents, the laws of conservation of energy and linear momentum, Eqs. (4.10a) and (4.10b), are preserved by the temporal invariance of U, Eq. (4.8a), and P^i, Eq. (4.8b). Meanwhile, conservation of angular momentum follows from the symmetry $T^{\alpha\beta} = T^{\beta\alpha}$ [19, 20] of the energy–momentum tensor, Eq. (4.4) .

The equations of motion for the macroscopic fields, Eqs. (3.10)–(3.12) and (3.16) contain sources that we now add to our homogeneous energy–momentum formalism. Recognizing that sources affect the conservation of energy and momentum, we require the sources to be perturbative. We form scalar products of a field with the equations of motion for the fields and combine the results to obtain the energy continuity equation

$$\frac{\partial}{\partial \bar{x}_0}\left[\frac{1}{2}(\mathbf{\Pi}^2 + \mathbf{B}^2)\right] + \nabla\cdot(\mathbf{B}\times\mathbf{\Pi}) = \frac{n\mathbf{J}}{c}\cdot\mathbf{\Pi} + \frac{\nabla n}{n}\cdot(\mathbf{B}\times\mathbf{\Pi}) \tag{4.12}$$

and the momentum continuity equation

$$\frac{\partial}{\partial \bar{x}_0}(\mathbf{B}\times\mathbf{\Pi})+\nabla\cdot\mathbf{W}=\rho\mathbf{\Pi}+\mathbf{B}\times\frac{n\mathbf{J}}{c}+\mathbf{\Pi}^2\frac{\nabla n}{n}. \quad (4.13)$$

Then, the inhomogeneous tensor continuity equation is

$$\bar{\partial}_\beta T^{\alpha\beta}=f^\alpha, \quad (4.14)$$

where

$$f^\alpha=\left(\frac{n\mathbf{J}}{c}\cdot\mathbf{\Pi}+\frac{\nabla n}{n}\cdot(\mathbf{B}\times\mathbf{\Pi}),\rho\mathbf{\Pi}+\mathbf{B}\times\frac{n\mathbf{J}}{c}+\mathbf{\Pi}^2\frac{\nabla n}{n}\right) \quad (4.15)$$

is the generalized force four-vector [2]. The inhomogeneous electromagnetic continuity equations give a general indication of the effect of sources, but they must be used cautiously. For example, the gradient of the refractive index must be sufficiently small that reflections can be neglected [2]. The presence of a charges and currents moving freely in a continuous medium has been accepted here as a historical imperative and any forces associated with the charges and currents should be regarded as perturbative.

V. SUMMARY

We have recast classical continuum electrodynamics into a region of space in which the speed of light is c/n, instead of c, and derived equations of motion for the macroscopic electromagnetic fields from the electromagnetic Lagrangian density. The success of a new physical theory is often gauged by its ability to resolve previously intractable problems. We presented a one-line derivation of the total energy–momentum tensor that demonstrates that the new representation is consistent with the continuum form of the laws of conservation of total energy and total momentum.

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